

Some remarks on P , P_0 , B and B_0 tensors ^{*}

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Abstract

Recently, Song and Qi extended the concept of P , P_0 and B matrices to P , P_0 , B and B_0 tensors, obtained some properties about these tensors, and proposed many questions for further research. In this paper, we answer three questions mentioned as above and obtain further results about P , P_0 , B and B_0 tensors.

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1 Introduction

Since the work of Lim [3] and Qi [6], the study of tensors (and hypergraphs) and their various applications has attracted much attention and interest.

A real m th order n -dimensional tensor (hypermatrix) $\mathbb{A} = (a_{i_1 i_2 \dots i_m})$ is a multi-array of real entries $a_{i_1 i_2 \dots i_m}$, where $i_j \in [n] = \{1, 2, \dots, n\}$ for $j \in [m]$. Denote the set of all real m th order n -dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m . Let $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$, if the entries $a_{i_1 i_2 \dots i_m}$ are invariant under any permutation of their indices, then \mathbb{A} is called a symmetric tensor. Denote the set of all real m th order n -dimensional symmetric tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$.

Let $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$ and $x \in \Re^n$. Then $\mathbb{A}x^m$ is a homogeneous polynomial of degree m , defined by

$$\mathbb{A}x^m = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \dots x_{i_m}.$$

A tensor $\mathbb{A} \in T_{m,n}$ is called positive semi-definite if for any vector $x \in \Re^n$, $\mathbb{A}x^m \geq 0$, and $\mathbb{A} \in T_{m,n}$ is called positive definite if for any nonzero vector $x \in \Re^n$, $\mathbb{A}x^m > 0$. Clearly, if m is odd, there is no nontrivial positive semi-definite tensors.

It is well known that P matrices and P_0 matrices which were first studied systematically by Fiedler and Pták [2] have a long history and wide applications in linear complementarity

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problems, variational inequalities and nonlinear complementarity problems and so on. In 2001, Pena proposed and studied B matrices [4], obtained many nice properties and applications of such matrices [4, 5], and proved that the class of B matrices is a subclass of P matrices [4]. Recently, Song and Qi extended the concept of P , P_0 and B matrices to P , P_0 , B and B_0 tensors, obtained some nice properties about these tensors in [7, 8], and they also proposed many questions for further research.

Let $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$ and $x \in C^n$. Then $\mathbb{A}x^{m-1}$ is a vector in C^n with its i th components as

$$(\mathbb{A}x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}. \quad (1.1)$$

for $i \in [n]$.

Definition 1.1. ([8], Definition 2) Let $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$. We say that \mathbb{A} is

- (a) a P_0 tensor if for any nonzero vector $x \in \Re^n$, there exists $i \in [n]$ such that $x_i \neq 0$ and $x_i(\mathbb{A}x^{m-1})_i \geq 0$;
- (b) a P tensor if for any nonzero vector $x \in \Re^n$, $\max_{i \in [n]} x_i(\mathbb{A}x^{m-1})_i > 0$.

Definition 1.2. ([8], Definition 3) Let $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$. We say that \mathbb{A} is a B tensor if for all $i \in [n]$, $\sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m} > 0$ and $\frac{1}{n^{m-1}}(\sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m}) > a_{i j_2 \dots j_m}$ for all $(j_2, \dots, j_m) \neq (i, \dots, i)$. We say that \mathbb{A} is a B_0 tensor if for all $i \in [n]$, $\sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m} \geq 0$ and

$$\frac{1}{n^{m-1}}(\sum_{i_2, \dots, i_m=1}^n a_{i i_2 \dots i_m}) \geq a_{i j_2 \dots j_m} \text{ for all } (j_2, \dots, j_m) \neq (i, \dots, i).$$

In [6], the definitions of eigenvalues, H -eigenvalues, E -eigenvalues and Z -eigenvalues of tensors in $S_{m,n}$ were proposed. In [8], Song and Qi extended there concepts to tensors in $T_{m,n}$, and they obtain some properties about P and P_0 tensors as follows.

Theorem 1.3. ([8], Theorem 1) Let $\mathbb{A} \in T_{m,n}$ be a $P(P_0)$ tensor. Then

- (1) When m is even, all of its H -eigenvalues and Z -eigenvalues of \mathbb{A} are positive (non-negative).
- (2) A symmetric tensor is a $P(P_0)$ tensor if and only if it is positive (semi-)definite.
- (3) There does not exist an odd order symmetric P tensor.
- (4) If an odd order non-symmetric P tensor exists, then it has no Z -eigenvalues.
- (5) An odd order P_0 tensor has no nonzero Z -eigenvalues.

Based on the result (3) of Theorem 1.3, Song and Qi proposed the following Question 1.4.

Question 1.4. ([8], Question 1) Is there an odd order non-symmetric P tensor? Is there an odd order nonzero non-symmetric P_0 tensor?

It is well known that each B matrix is a P matrix, namely, when $m = 2$, each B tensor is a P tensor. In [8], the authors noted that when m is odd, in general, a $B(B_0)$ tensor is not a $P(P_0)$ tensor. For example, let $a_{i \dots i} = 1$ and $a_{i_1 \dots i_m} = 0$ otherwise. Then $\mathbb{A} = (a_{i_1 \dots i_m})$ is the identity tensor. But when m is odd, the identity tensor is a B tensor, not a P or P_0 tensor. Thus they proposed the following Question 1.5.

Question 1.5. ([8], Question 4) *When $m \geq 4$ and is even, is a $B(B_0)$ tensor a $P(P_0)$ tensor?*

In [7], the authors proved that an even order symmetric B tensor is positive definite, and an even order symmetric B_0 tensor is positive semi-definite. Combining the result (2) of Theorem 1.3, we know that an even order symmetric B tensor is a P tensor and an even order symmetric B_0 tensor is a P_0 tensor. Further, the author proposed the following question.

Question 1.6. ([7], Question 2) *Can we show that an even order non-symmetric B tensor is a P tensor and an even order non-symmetric B_0 tensor is a P_0 tensor?*

In this paper, we answer the above three questions and obtain further results about P , P_0 , B and B_0 tensors.

2 Some remarks on P and P_0 tensors

In this section, we show that there does not exist an odd order P tensor and there exist odd order nonzero non-symmetric P_0 tensors, thus we give the answer of Question 1.4.

Proposition 2.1. *There does not exist an odd order P tensor.*

Proof. Let m be odd and $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$ be an odd order P tensor. For some given $k \in [n]$, we take nonzero vector $x = (0, \dots, 0, u, 0, \dots, 0) = (x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n)$, where $x_k = u (\neq 0)$ and $x_i = 0$ for $i \neq k$. Since for any $i \in [n]$, by (1.1), we have

$$(\mathbb{A}x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m} = a_{ik \dots k} u^{m-1},$$

then

$$x_i (\mathbb{A}x^{m-1})_i = \begin{cases} 0, & \text{if } i \neq k; \\ a_{kk \dots k} u^m, & \text{if } i = k. \end{cases}$$

Since \mathbb{A} is a P tensor, we have $\max_{i \in [n]} x_i (\mathbb{A}x^{m-1})_i = a_{kk \dots k} u^m > 0$, then $a_{kk \dots k} \neq 0$. Noting that m is odd, thus $a_{kk \dots k} > 0$ when we take $u > 0$ and $a_{kk \dots k} < 0$ when we take $u < 0$ by $a_{kk \dots k} u^m > 0$, it is a contradiction. Thus there does not exist an odd order P tensor. \square

In [8], the authors proved that there does not exist an odd order symmetric P tensor by the theory of eigenvalues of tensors, now we can obtain the result by proposition 2.1 directly.

Corollary 2.2. (1) ([8], (3) of Theorem 1) *There does not exist an odd order symmetric P tensor.*

(2) *There does not exist an odd order non-symmetric P tensor.*

Proposition 2.3. *There exist odd order nonzero non-symmetric P_0 tensors.*

Proof. Let m be odd. Now we show the result by the following two cases.

Case 1: n is even.

Let $n = 2k \geq 2$. Take $a_{i(k+i) \dots i} = 1$, $a_{(k+i) \dots i} = -2$ for any $i \in [k]$ and $a_{i_1 \dots i_m} = 0$ for others $(i_1, \dots, i_m) \in [n]^m$. Clearly, $\mathbb{A} = (a_{i_1 \dots i_m})$ is an odd order nonzero non-symmetric tensor. Now we show that \mathbb{A} is a P_0 tensor.

For any nonzero vector $x = (x_1, x_2, \dots, x_{2k})^T \in \Re^{2k}$, noting that

$$x_i(\mathbb{A}x^{m-1})_i = x_i^{m-1}x_{k+i}, \quad x_{k+i}(\mathbb{A}x^{m-1})_{k+i} = -2x_i^{m-1}x_{k+i} \text{ for any } i = 1, \dots, k,$$

then by $x \neq 0$, there exists some $j \in [k]$ such that $x_j \neq 0$ or $x_{k+j} \neq 0$, and

$$x_j(\mathbb{A}x^{m-1})_j \geq 0 \text{ or } x_{k+j}(\mathbb{A}x^{m-1})_{k+j} \geq 0.$$

Thus \mathbb{A} is a P_0 tensor by the definition of P_0 tensor.

Case 2: n is odd.

Let $n = 2k + 1 \geq 3$. Take $a_{i(k+i)i\dots i} = 1, a_{(k+i)ii\dots i} = -2$ for $i \in [k - 1]$, $a_{k(2k)(2k+1)k\dots k} = 1, a_{(2k)(2k+1)k\dots k} = -2, a_{(2k+1)(2k)k\dots k} = 4$, and $a_{i_1\dots i_m} = 0$ for others $(i_1, \dots, i_m) \in [n]^m$. Clearly, $\mathbb{A} = (a_{i_1\dots i_m})$ is an odd order nonzero non-symmetric tensor. Now we show that \mathbb{A} is a P_0 tensor.

For any nonzero vector $x = (x_1, x_2, \dots, x_{2k+1})^T \in \Re^{2k+1}$, noting that

$$x_i(\mathbb{A}x^{m-1})_i = x_i^{m-1}x_{k+i}, \quad x_{k+i}(\mathbb{A}x^{m-1})_{k+i} = -2x_i^{m-1}x_{k+i} \text{ for any } i = 1, \dots, k - 1,$$

and $x_k(\mathbb{A}x^{m-1})_k = x_k^{m-2}x_{2k}x_{2k+1}$, $x_{2k}(\mathbb{A}x^{m-1})_{2k} = -2x_k^{m-2}x_{2k}x_{2k+1}$, $x_{2k+1}(\mathbb{A}x^{m-1})_{2k+1} = 4x_k^{m-2}x_{2k}x_{2k+1}$, then by $x \neq 0$, we can complete the proof by the following two cases.

Subcase 2.1: there exists some $j \in [k - 1]$ such that $x_j \neq 0$ or $x_{k+j} \neq 0$.

It is obvious that \mathbb{A} is a P_0 tensor by $x_j(\mathbb{A}x^{m-1})_j \geq 0$ or $x_{k+j}(\mathbb{A}x^{m-1})_{k+j} \geq 0$ and the definition of P_0 tensor.

Subcase 2.2: $x_k \neq 0$ or $x_{2k} \neq 0$ or $x_{2k+1} \neq 0$.

It is obvious that \mathbb{A} is a P_0 tensor by the definition of P_0 tensor and the facts that $x_k(\mathbb{A}x^{m-1})_k \geq 0$ or $x_{2k}(\mathbb{A}x^{m-1})_{2k} \geq 0$ or $x_{2k+1}(\mathbb{A}x^{m-1})_{2k+1} \geq 0$.

Combining the above arguments, \mathbb{A} is an odd order nonzero non-symmetric P_0 tensor. \square

Remark 2.4. Noting that whether “ m is odd” or not does not change the fact that \mathbb{A} is a nonzero non-symmetric P_0 tensor. Thus the proof of Proposition 2.3 also give an example of an even order nonzero non-symmetric P_0 tensor.

Clearly, Propositions 2.1 and 2.3 answer Question 1.4 (Question 1 in [8]).

3 The relationship between $B(B_0)$ and $P(P_0)$ tensors

In this section, we answer Questions 1.5 and 1.6, obtain some further results about $B(B_0)$ tensors and $P(P_0)$ tensors.

Proposition 3.1. When $m \geq 4$ and is even, in general, a $B(B_0)$ tensor is not a $P(P_0)$ tensor.

Proof. Let $m(\geq 4)$ be even and $n(\geq 3)$ be a positive integer, $\mathbb{A} = (a_{i_1\dots i_m})$ where $a_{11\dots 1} = h + 1 > 1, a_{1i_2\dots i_m} = h > 0, a_{ii\dots i} = 1$ and $a_{i1i_2\dots i} = b$ for any $i \in \{2, \dots, n\}$, and $a_{i_1i_2\dots i_m} = 0$ for others.

It is easy to check that \mathbb{A} is an even order non-symmetric B tensor when $b = -\frac{1}{2}$ and \mathbb{A} is an even order non-symmetric B_0 tensor when $b = -1$ by the definitions of $B(B_0)$ tensor.

Now for any nonzero vector $x = (x_1, \dots, x_n)^T \in \Re^n$, we have

$$x_i(\mathbb{A}x^{m-1})_i = \begin{cases} x_1^m + hx_1(x_1 + \dots + x_n)^{m-1}, & \text{if } i = 1; \\ x_i^m + bx_1^2x_i^{m-2}, & \text{if } i \in \{2, \dots, n\}. \end{cases}$$

It is easy to see that there exist real numbers h, x_1, \dots, x_n with $x_i (\mathbb{A}x^{m-1})_i < 0$ for all $i \in [n]$ and $x_i \neq 0$. For example, we take $x_1 = -3, x_2 = \dots = x_n = 2, h > (\frac{3}{2n-5})^{m-1}$, we have $x_i (\mathbb{A}x^{m-1})_i < 0$ for all $i \in [n]$ and $x_i \neq 0$. Then \mathbb{A} is not a $P(P_0)$ tensor by the definitions of $P(P_0)$ tensor. \square

Remark 3.2. Clearly, Proposition 3.1 answers Questions 1.5 and 1.6 (Question 4 in [8] and Question 2 in [7]). Furthermore, by Proposition 3.1 and some known results, we know that a $B(B_0)$ tensor is not a $P(P_0)$ tensor in general. That is to say, the cases of $m \geq 3$ (hypermatrices) is different from the case of $m = 2$ (matrices) completely.

Let $\mathbb{A} \in T_{m,n}$. If all of the off-diagonal entries of \mathbb{A} are non-positive, i.e., $a_{i_1 \dots i_m} \leq 0$ when $(i_1, \dots, i_m) \neq (i, \dots, i)$, then \mathbb{A} is called a Z tensor ([9]). In the following, we will show if an even order Z tensor \mathbb{A} is a $B(B_0)$ tensor, then \mathbb{A} is a $P(P_0)$ tensor.

We call \mathbb{A} is diagonally dominated if for all $i \in [n]$,

$$a_{i \dots i} \geq \sum \{|a_{ii_2 \dots i_m}| : (i_2, \dots, i_m) \neq (i, \dots, i), i_j \in [n], j = 2, \dots, m\};$$

and we call \mathbb{A} is strictly diagonally dominated if for all $i \in [n]$,

$$a_{i \dots i} > \sum \{|a_{ii_2 \dots i_m}| : (i_2, \dots, i_m) \neq (i, \dots, i), i_j \in [n], j = 2, \dots, m\}.$$

It was proved in [9] that a diagonally dominated Z tensor is an M tensor, and a strictly diagonally dominated Z tensor is a strong M tensor ([1, 9]), where strong M tensors are called nonsingular tensors ([1]), and Laplacian tensors of uniform hypergraphs which is defined as a natural extension of Laplacian matrices of graphs are M tensors.

In [8], the authors give the properties of a $B(B_0)$ tensor under the condition that it is a Z tensor as follows.

Theorem 3.3. ([8], Theorem 8) *Let $\mathbb{A} = (a_{i_1 i_2 \dots i_m}) \in T_{m,n}$ be a Z tensor. Then the following properties are equivalent:*

- (i) \mathbb{A} is $B(B_0)$ tensor;
- (ii) for each $i \in [n]$, $\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}$ is positive (nonnegative);
- (iii) \mathbb{A} is strictly diagonally dominant (diagonally dominated).

Theorem 3.4. *Let m be even, $\mathbb{A} \in T_{m,n}$ be a strictly diagonally dominated (diagonally dominated) tensor, then \mathbb{A} is a $P(P_0)$ tensor.*

Proof. Let $\mathbb{A} \in T_{m,n}$ be a strictly diagonally dominated tensor. Then for any nonzero vector $0 \neq x \in \Re^n$, without loss of generality, we assume that $|x_1| = \max_{i \in [n]} \{|x_i|\}$, then $|x_1| \neq 0$. By the definition of strictly diagonally dominated and m is even, we have

$$\begin{aligned} x_1 (\mathbb{A}x^{m-1})_1 &= a_{1 \dots 1} |x_1|^m + \sum_{i_2, \dots, i_m=1, (i_2, \dots, i_m) \neq (1, \dots, 1)}^n a_{1i_2 \dots i_m} x_1 x_{i_2} \dots x_{i_m} \\ &\geq \left(a_{1 \dots 1} - \sum_{i_2, \dots, i_m=1, (i_2, \dots, i_m) \neq (1, \dots, 1)}^n |a_{1i_2 \dots i_m}| \right) |x_1|^m \\ &> 0. \end{aligned}$$

Thus \mathbb{A} is a P tensor by the definition of P tensor.

If $\mathbb{A} \in T_{m,n}$ is a diagonally dominated tensor, the proof is similar, we omit it. \square

By Theorem 3.4 and the result (2) of Theorem 1.3, we obtain the following result immediately.

Corollary 3.5. ([7], Theorem 3) *Let m be even and $\mathbb{A} \in S_{m,n}$. If \mathbb{A} is diagonally dominated, then \mathbb{A} is positive semi-definite. If \mathbb{A} is strictly diagonally dominated, then \mathbb{A} is positive definite.*

By Theorems 3.3 and 3.4, we obtain the following result directly.

Theorem 3.6. *Let m be even, $\mathbb{A} \in T_{m,n}$ be a Z tensor. If \mathbb{A} is a $B(B_0)$ tensor, then \mathbb{A} is a $P(P_0)$ tensor.*

Remark 3.7. *Let $\mathbb{A} \in T_{m,n}$ be a Z tensor. If \mathbb{A} is a B tensor, in general, \mathbb{A} is not positive definite. For example, let $b > 0, h > 0$ and $\mathbb{A} = (a_{i_1 \dots i_m})$, where $a_{11 \dots 1} = b + 1$, $a_{12 \dots 2} = -b$, $a_{ii \dots i} = 1$ for all $i \in \{2, \dots, n\}$, and $a_{i_1 i_2 \dots i_m} = 0$ for others. Clearly, \mathbb{A} is a strictly diagonally dominated Z tensor, namely, \mathbb{A} is both a Z tensor and a B tensor by Theorem 3.3. Then for any vector $x = (x_1, \dots, x_n)^T \in \Re_n$, we have $\mathbb{A}x^m = \sum_{i=1}^n x_i^m + bx_1^m - bx_1x_2^{m-1}$. Now we take $x_1 = 1, x_2 = \dots = x_n = h$, thus $\mathbb{A}x^m = (b + 1) + (n - 1)h^m - bh^{m-1}$. Clearly, there exist positive real numbers b and h such that $\mathbb{A}x^m < 0$, then \mathbb{A} is not positive definite. But by Theorem 3.6, \mathbb{A} is a P tensor, it implies that a P tensor, in general, is not positive definite.*

4 Decomposition of $B(B_0)$ tensors

In [7], the authors show that a symmetric $B(B_0)$ tensor can always be decomposed to the sum of a strictly diagonally dominated (diagonally dominated) symmetric Z tensor and several positive multiples of partially all one tensors. When the order is even, this result implies that a symmetric $B(B_0)$ tensor is positive definite (positive semi-definite), namely, a symmetric $P(P_0)$ tensor. In this section, we weaken the condition “symmetric” and obtain the generalized results by using some similar technique in [7].

This concept of “a principal sub-tensor” was first introduced and used in [6] for symmetric tensor as follows.

Definition 4.1. ([6]) *A tensor $\mathbb{C} \in T_{m,r}$ is called a principal sub-tensor of a tensor $\mathbb{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$ ($1 \leq r \leq n$) if there is a set J that composed of r elements in $[n]$ such that*

$$\mathbb{C} = (a_{i_1 \dots i_m}), \text{ for all } i_1, i_2, \dots, i_m \in J.$$

For convenient, we denote by \mathbb{A}_r^J the principal sub-tensor of a tensor $\mathbb{A} \in T_{m,n}$ such that the entries of \mathbb{A}_r^J are indexed by $J \subseteq [n]$ with $|J| = r$ for $1 \leq r \leq n$. It is clear that when $r = 1$, the principal sub-tensors are just the diagonal entries.

Definition 4.2. ([7]) *Suppose that $A \in S_{m,n}$ has a principal sub-tensor A_r^J with $J \subseteq [n]$ with $|J| = r$ ($1 \leq r \leq n$) such that all the entries of A_r^J are one, and all the other entries of A are zero. Then A is called a partially all one tensor, and denoted by \mathcal{E}^J . If $J = [n]$, then we denote \mathcal{E}^J simply by \mathcal{E} and call it an all one tensor.*

When m is even, if we denote by x_J the r -dimensional sub-vector of a vector $x \in \Re^n$, with the components of x_J indexed by J , then for any $x = (x_1, \dots, x_n)^T \in \Re^n$, we have

$$\mathcal{E}^J x^m = \left(\sum \{x_j : j \in J\} \right)^m \geq 0. \quad (4.1)$$

Thus an even order partially all one tensor is positive semi-definite.

Let S_k be the set of all permutations σ of the integers $1, 2, \dots, k$, and symbol

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.3. *Let $\mathbb{A} = (a_{i_1 \dots i_m}) \in T_{m,n}$ be a tensor such that $a_{i_1 \dots i_m} = a_{\sigma(i_1) \dots \sigma(i_m)}$ for any $\sigma \in S_m$ whenever $a_{i_1 \dots i_m} > 0$ and $\delta_{i_1 \dots i_m} = 0$.*

(1) *If \mathbb{A} is a B_0 tensor, then either \mathbb{A} is a diagonally dominated Z tensor itself, or we have*

$$\mathbb{A} = \mathbb{M} + \sum_{k=1}^s h_k \mathcal{E}^{J_k}, \quad (4.2)$$

where \mathbb{M} is a diagonally dominated Z tensor, s is a positive integer, $h_k > 0$ and $J_k \subseteq [n]$, for $k = 1, \dots, s$, and $J_s \subsetneq J_{s-1} \subsetneq \dots \subsetneq J_1$.

(2) *If \mathbb{A} is a B tensor, then either \mathbb{A} is a strictly diagonally dominated Z tensor itself, or we have (4.2) where \mathbb{M} is a strictly diagonally dominated Z tensor.*

Proof. Now we show (1) holds. Define $J(\mathbb{A}) \subseteq [n]$ as

$$J(\mathbb{A}) = \{i \in [n] : \text{there is at least one positive off-diagonal entry in the } i\text{th row of } \mathbb{A}\}.$$

Case 1: $J(\mathbb{A}) = \emptyset$.

Then \mathbb{A} is a Z tensor, thus a diagonally dominated Z tensor by Theorem 3.3. The result holds.

Case 2: $J(\mathbb{A}) \neq \emptyset$.

Let $\mathbb{A}_1 = \mathbb{A}$. For each $i \in J(\mathbb{A}_1) = J_1$, let d_i be the value of the largest off-diagonal entry in the i th row of \mathbb{A}_1 . Let

$$h_1 = \min\{d_i : i \in J_1\}, \quad |J_1| = r \leq n.$$

Then $h_1 > 0$.

By the definitions of \mathbb{A}_1 and J_1 , we know that for the indices i, i_2, \dots, i_m with $a_{ii_2 \dots i_m} > 0$ and $\delta_{ii_2 \dots i_m} = 0$, we have

$$i, i_2, \dots, i_m \in J_1.$$

This implies that for the indices i_1, i_2, \dots, i_m with $\delta_{i_1 i_2 \dots i_m} = 0$, if there exists some $j \in [m]$ such that $i_j \notin J_1$, then $a_{i_1 i_2 \dots i_m} \leq 0$.

Now we consider

$$\mathbb{A}_2 = \mathbb{A}_1 - h_1 \mathcal{E}^{J_1} = (a_{i_1 i_2 \dots i_m}^{(2)}).$$

It is clear that $a_{i_1 \dots i_m}^{(2)} = a_{\sigma(i_1) \dots \sigma(i_m)}^{(2)}$ for any $\sigma \in S_m$ whenever $a_{i_1 \dots i_m}^{(2)} > 0$ and $\delta_{i_1 \dots i_m} = 0$. Now we show that \mathbb{A}_2 is still a B_0 tensor.

By the definition of \mathcal{E}^{J_1} , we have

$$a_{i_1 i_2 \dots i_m}^{(2)} = \begin{cases} a_{i_1 i_2 \dots i_m} - h_1, & \text{if } i_1, i_2, \dots, i_m \in J_1; \\ a_{i_1 i_2 \dots i_m}, & \text{if there exists some } i_j \notin J_1 \text{ where } j \in [m]. \end{cases}$$

Thus for the index $i \in J_1$, there exist indices $j_2, \dots, j_m \in J_1$ with $\delta_{ij_2 \dots j_m} = 0$ such that $a_{ij_2 \dots j_m}^{(2)} = a_{ij_2 \dots j_m} - h_1 \geq 0$, then by \mathbb{A} is a B_0 tensor, we have

$$\begin{aligned}
& \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}^{(2)} \right) \\
&= \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m \in J_1} (a_{ii_2 \dots i_m} - h_1) + \sum_{\text{there exists some } i_j \notin J_1 \text{ where } 2 \leq j \leq m} a_{ii_2 \dots i_m} \right) \\
&= \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} \right) - \frac{n^{m-1}}{n^{m-1}} h_1 \\
&\geq \max\{a_{ij_2 \dots j_m} : \delta_{ij_2 \dots j_m} = 0\} - h_1 \\
&= \max\{a_{ij_2 \dots j_m}^{(2)} : \delta_{ij_2 \dots j_m} = 0\} \\
&\geq 0,
\end{aligned}$$

and for the other index $i \notin J_1$, we know for any indices $j_2, \dots, j_m \in [n]$ with $\delta_{ij_2 \dots j_m} = 0$, $a_{ij_2 \dots j_m}^{(2)} = a_{ij_2 \dots j_m} \leq 0$ and $a_{ii \dots i}^{(2)} = a_{ii \dots i}$, then by \mathbb{A} is a B_0 tensor, we have

$$\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}^{(2)} = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} \geq 0,$$

and

$$\begin{aligned}
& \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}^{(2)} \right) \\
&= \frac{1}{n^{m-1}} \left(\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} \right) \\
&\geq \max\{a_{ij_2 \dots j_m} : \delta_{ij_2 \dots j_m} = 0\} \\
&= \max\{a_{ij_2 \dots j_m}^{(2)} : \delta_{ij_2 \dots j_m} = 0\}.
\end{aligned}$$

Therefore $\mathbb{A}_2 = (a_{i_1 \dots i_m}^{(2)})$ is still a B_0 tensor such that $a_{i_1 \dots i_m}^{(2)} = a_{\sigma(i_1) \dots \sigma(i_m)}^{(2)}$ for any $\sigma \in S_m$ whenever $a_{i_1 \dots i_m}^{(2)} > 0$ and $\delta_{i_1 \dots i_m} = 0$.

We now replace \mathbb{A}_1 by \mathbb{A}_2 , and repeat this process. We see that

$$J_2 = J(\mathbb{A}_2) = \{i \in [n] : \text{there is at least one positive off-diagonal entry in the } i\text{th row of } \mathbb{A}_2\}.$$

is a proper subset of $J(\mathbb{A}_1) = J_1$. Repeat this process until $J_{s+1} = J(\mathbb{A}_{s+1}) = \emptyset$. Let $\mathbb{M} = \mathbb{A}_{s+1}$. We see that (4.2) holds. It is obvious that s is a positive integer, $h_k > 0$ and $J_k \subseteq [n]$, for $k = 1, \dots, s$, and $J_s \subsetneq J_{s-1} \subsetneq \dots \subsetneq J_1$. This proves (1).

The proof of (2) is similar to the proof of (1), so we omit it. \square

By Theorem 4.3 and the result (2) of Theorem 1.3, we can obtain the following results immediately.

Corollary 4.4. ([7], Theorem 4)

- (1) An even order symmetric B_0 tensor is a P_0 tensor and positive semi-definite.
- (2) An even order symmetric B tensor is a P tensor and positive definite.

Proof. Now we only show (1) holds, the proof of (2) is similar, we omit it.

Let m be even and $\mathbb{A} = (a_{i_1 \dots i_m})$ be a symmetric B_0 tensor. By Theorem 4.3, if \mathbb{A} itself is a diagonally dominated symmetric Z tensor, then it is a P_0 tensor and positive semi-definite by Theorem 3.6. Otherwise, (4.2) holds with $s > 0$. Let $x = (x_1, \dots, x_n)^T \in \Re^n$. Then by (4.1), (4.2) and \mathbb{M} is a diagonally dominated symmetric Z tensor, we have

$$\mathbb{B}x^m = \mathbb{M}x^m + \sum_{k=1}^s h_k \mathcal{E}^{J_k} x^m = \mathbb{M}x^m + \sum_{k=1}^s h_k \left(\sum \{x_j : j \in J_k\} \right)^m \geq 0.$$

This implies (1) holds by the definition of positive semi-definite and result (2) of Theorem 1.3. \square

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